Creation and dynamics of vortex tubes in three-dimensional turbulence

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We examine the possibility of a blowup of the vorticity due to self-stretching and mechanisms for its prevention. We first estimate directly from the Navier-Stokes equations the length scale of coherence in the direction of the vortex lines to be of the order of the Kolmogorov length. Alignment of vortex lines is seen to be a viscous phenomenon and may prevent some scenarios for blowup. Next we derive equations for the curvature and torsion of vortex lines. We show that the same stretching that amplifies the vorticity also tends to straighten out the vortex lines. Then we show that in well-aligned vortex tubes, the self-stretching rate of the vorticity is proportional to the ratio of the vorticity and the radius of curvature. Thus blowup of the vorticity in such tubes can be prevented by the growth of the vorticity being balanced by the straightening of the vortex lines. Implications for vorticity-strain alignment and the scaling theory of turbulence are noted. Finally, we examine the effects of viscous diffusion on the vorticity field and see how viscosity can lead to organization and alignment of vortex lines.

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I. INTRODUCTION

In direct simulations of strong isotropic turbulence [1,2] one observes that regions of high vorticity are organized into a collection of well-aligned structures and in particular tubelike structures. It is our intention to investigate this phenomenon in relation to two fundamental problems in fluid mechanics. The first is the question of the regularity of the solutions of the initial value problem of the Navier-Stokes equation [3], and the second is the issue of scaling exponents in turbulence [4,5]. In this paper we discuss the structure of high vorticity regions and its connection to these two problems.

The vorticity field $\boldsymbol{\omega}(x,t)$ in a fluid is the curl of the velocity filed $\mathbf{u}(x,t)$, $\boldsymbol{\omega}(x,t)=\nabla\times\mathbf{u}(x,t)$. The velocity field $\mathbf{u}(x,t)$ is assumed to attain its largest variations (which are denoted by U) on a scale L (the "integral scale"). The Reynolds number Re is defined as UL/ν, where ν is the viscosity. In incompressible flow the equation of motion of the vorticity reads

$$D_t\boldsymbol{\omega}\equiv\frac{\partial\boldsymbol{\omega}}{\partial t}+\mathbf{u}\cdot\nabla\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}+\nu\nabla^2\boldsymbol{\omega}\ . \tag{1}$$

The magnitude of $\boldsymbol{\omega}(x,t)$ is denoted by w, $w\equiv|\boldsymbol{\omega}(x,t)|$. Its equation of motion will be important for much of our analysis below. To obtain this equation, multiply Eq. (1) by $\boldsymbol{\omega}$ and divide by w. The resulting equation reads (for $w\neq0$)

$$\frac{\partial w}{\partial t}+\mathbf{u}\cdot\nabla w-\nu\nabla^2 w=(\alpha-\nu|\nabla\xi|^2)w\ , \tag{2}$$

where $\xi=\boldsymbol{\omega}/w$ and

$$\alpha=\xi\cdot(\xi\cdot\nabla)\mathbf{u}\ . \tag{3}$$

Thus α is the diagonal component in the ξ direction of the strain tensor, which is the symmetric part of $\nabla\mathbf{u}$.

The strain can be expressed as a nonlocal linear operator of the vorticity, the vorticity acting as a source for the strain (the expression can be found in Sec. IV). Since the local contribution to the strain is proportional to the local vorticity, the first term on the right-hand side of (2) is potentially quadratic. A quadratic nonlinearity could cause a finite time singularity. It is not known whether a singularity can appear spontaneously in solutions of the Navier-Stokes equations. The phenomenon where the stretching rate α of the vorticity is indeed induced locally by the vorticity, thus leading to a rapid growth of the vorticity magnitude, is called *self-stretching*.

Can the vorticity field be organized so that a finite time blowup is avoided? In our analysis below we examine aspects of the structure both of interest in themselves and of relevance to this process. In the first part of the paper we show that indeed such an organization is possible and even plausible and see, conversely, what the implications for the structure of the vorticity are if we assume that a blowup does not occur. We also apply the results of our analysis to vortex tube structures of the type seen in numerical simulations.

The picture of vortex tubes used in this paper is of *coherent bundles of vortex lines,* or those lines whose tangent at any point is in the direction of the vorticity field. In other words, we study the spatial structure of the vorticity *direction* as the quantity of interest. We will therefore examine the vector field ξ defined above and form length scales from the rates of change of ξ in space, proposing the local direction gradient as a measure of vortex tube width and the local curvature of the vortex lines as a measure of vortex tube curvature. These are the length scales most relevant to how the strain is induced by the vorticity and therefore for the nonviscous dynamics of the vorticity. Our analysis of the structure of vorticity should allow us to understand two length scales that characterize the vortex tubes: their width and their radius of curvature. In defining these length scales the most common approach is to look at isosurfaces or Gaussian profile widths of vorticity magnitude. Such a procedure, however, is ill defined. Rather we opt, in this paper, to define everything unambiguously in terms of local field variables.

For these scales to be relevant for defining vortex tubes we must first show that the vorticity field is indeed aligned. In Sec. II we bound the divergence of the vortex lines in terms of the dissipation; the bound is seen to stem from viscous effects. In terms of ξ this means that $|\nabla\xi|$ must be small. The value of $|\nabla\xi|^{-1}$ is a measure of the length scale over which directional coherence is maintained. We obtain a bound from below for the mean value (over a short time and a small ball) of λ_c in terms of quantities that are familiar to the students of turbulence. Indeed it turns out that the diameter of the tubes λ_c is not a new length in the theory of turbulence, but is just a local version of the Kolmogorov scale. This is the first testable prediction of the theory. This scale is seen as that at which the rate of amplification by strain is balanced by the rate of diffusion out of the tube. We suggest that this can be checked in numerical calculations. We note that in a recent study by Jiménez *et al.* [6] the diameter of vortex tubes is evaluated from the Gaussian profile width of the vorticity *magnitude* for Reynolds numbers from 35 to 170; their conclusion is that this width scales with the Kolmogorov length. In fact, this scale was proposed some time ago [7] for this width. This scale is then offered as an estimate of vortex tube width in terms of coherence of direction. We suggest that in scenarios in which vorticity blows up through convergence towards vortex line corners or cusps [8,9], the vorticity will be controlled by the viscosity at this scale.

The understanding of the other length scales involved, in particular, the radius of curvature, takes us, in Sec. III, to the Euler dynamics of the vortex structures. We will see that there exists a mechanism that is responsible for exponential growth of the radius of curvature of vortex lines at the same rate of growth as the vorticity magnitude w. Thus the picture that emerges is that in those regions in which the vorticity becomes intense there exists a process of alignment and straightening of the vortex lines, in agreement with the phenomenological observations of long vortex tubes in which the vortex lines are strongly aligned [10].

In Sec. IV we discuss a way to avoid a blowup due to the quadratic growth rate created by a persistent curvature of a vortex tube. We show that in the specific case of well-aligned vortices, the growth rate should be proportional to the ratio of the vorticity and the radius of curvature. By the results of Sec. III a blowup can be avoided if the curvature shrinks at the same rate as the vorticity grows.

In Sec. V we discuss the connection of the present topics to the scaling theory of turbulence. We reiterate that deviations from the Kolmogorov predictions, if they exist, can be due to the tendency towards the formation of locally two-dimensional structures [5]. Regrettably, we have to admit in this section that, although we have an interesting mechanism for such a tendency, we are still missing a very important link to the statistical theory of turbulence in that we do not know presently how to assess the contribution of the intense vorticity regions to overall averages. If these regions concentrate on fractal sets, as some authors have suggested, they may become irrelevant for the statistical averages and the Kolmogorov theory may be exact for sufficiently high Re. It is our feeling the numerical simulations can be helpful in assessing this issue and we make some comments in this direction.

In Sec. VI we examine more closely the effects of viscosity on vorticity and vortex structures. In the earlier sections we ascribed the alignment of vortex lines to the action of viscosity. Just how the viscous diffusion of the components of a vector field can achieve such an effect was left open however. We analyze the simplest case of diffusion of a vector field—that of two-component vectors on a line. Diffusion of vectors turns out to have non-trivial effects and we see how diffusion can cause alignment both by dissipating unaligned regions and by dynamically realigning vortex lines. We see that we expect alignment to be particularly strong near maxima in field strength.

In the Appendix we analyze the full three-dimensional case. Here the situation is much more complex. We exhibit the effect of viscosity on the vorticity magnitude and direction in the Frenet frame and see that our analysis of the two-dimensional vector case allows us to understand much of what is happening; new three-dimensional effects include new alignment effects. We see that we expect alignment to be particularly strong in strong vortex tubes.

II. LOCAL ALIGNMENT OF VORTEX LINES: VISCOSITY AT WORK

In this section we will show that vortex lines tend to align due to the action of viscosity. There exists a mean scale characterizing the directional coherence, which we denote by λ_c. We will interpret this scale below as the "diameter" of the vortex tubes.

A. Definition of the directional coherence length

Our definition of the length λ_c follows from considering the spatial derivatives of the direction of vorticity. Defining the vorticity direction vector as $\xi=\omega/w$, we examine the scale over which this unit vector changes its direction significantly, which is measured by $|\nabla\xi|^{-1}$. Clearly, this quantity, which has the dimension of a length, is well defined everywhere in the fluid where $w\neq 0$ and can be measured at every such spatial point in numerical simulations. We are interested here in regions of stronger vorticity, so these points suffice. In order to be able to estimate this length at some point $\mathbf{x}$ from the equations of motion, we will average over a small ball around $\mathbf{x}$ which moves with the fluid and over an appropriate interval of time. Explicitly, we define

$$\frac{1}{\lambda_c}(\mathbf{x}) \equiv \frac{3}{4\pi r^3 t_r}\int_{|\mathbf{y}|\le r} d\mathbf{y}\int_{t_0}^{t_0+t_r} dt\,|\nabla\xi(\mathbf{x}+\mathbf{u}_0 t+\mathbf{y},t)|$$
$$\equiv \langle |\nabla\xi| \rangle_{r,t_r} \tag{4}$$

the average being over a ball B_r of some radius r around $\mathbf{x}$ moving with velocity $\mathbf{u}_0$, the mean velocity of the fluid in the ball

$$\mathbf{u}_0 = \frac{3}{4\pi r^3}\int_{|\mathbf{y}|\le r} \mathbf{u}(\mathbf{x}+\mathbf{y},t_0)d\mathbf{y}\ , \tag{5}$$

and over some short time t_r, starting at t_0. The choice of r and t_r will be made later. In regions of large vorticity, we can use the Cauchy-Schwartz inequality

$$\langle AB \rangle \le \langle A \rangle^{1/2} \langle B \rangle^{1/2}\ ,$$

taking $A = 1/\sqrt{w}$ and $B = \sqrt{w}\,|\nabla\xi|$ to write

$$\frac{1}{\lambda_c} \le \langle w|\nabla\xi|^2 \rangle_{r,t_r}^{1/2} \left\langle \frac{1}{w} \right\rangle_{r,t_r}^{1/2}\ . \tag{6}$$

B. Estimation of the coherence length

To estimate the first factor on the right-hand side of Eq. (6) we follow the methods of [4]. We start with the equation of motion for w, which has been written above as Eq. (2). To get averages of the type appearing in Eqs. (4)–(6) we employ a cutoff function $\phi(\mathbf{x},t)=\phi_0[(\mathbf{x}-\mathbf{x}_0-\mathbf{u}_0 t)/r]$, where the function ϕ_0 is not specified precisely and only needs to be such that $\phi_0(\mathbf{y})=1$ for $|\mathbf{y}|<\frac{1}{2}, \phi_0(\mathbf{y})=0$ for $|\mathbf{y}|>1$, and $\phi_0(\mathbf{y})$ is smoothed out sufficiently to have smooth derivatives. Thus ϕ is a time-dependent weight function, with support inside the moving ball of integration.

Multiplying Eq. (2) by ϕ and integrating over space we get

$$\nu\int d\mathbf{x}\, w|\nabla\xi|^2\phi = \int d\mathbf{x}\, \alpha w\phi - \frac{d}{dt}\int d\mathbf{x}\, w\phi$$
$$+\int d\mathbf{x}\left[\left[\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla-\nu\nabla^2\right]\phi\right] w\ . \tag{7}$$

At this point we have a choice of strategy. We can use Eq. (7) to bound the left-hand side in terms of initial data or in terms of evolving local quantities. The first strategy has been investigated in Ref. [3], with the result that $\langle w|\nabla\xi|^2\rangle_{r,t}$ is bounded in terms of initial data. The meaning of this result is that $|\nabla\xi|$ has to be small on the (local) average in regions of sufficiently high vorticity. This means that vortex lines align locally when w is large.

It turns out that these *a priori* bounds lead through (6) to a definition of length scale that is very small, perhaps unrealistically so. To achieve a physically more plausible length we follow now the second strategy of bounding $\langle w|\nabla\xi|^2\rangle_{r,t}$ in terms of locally evolving quantities. Taking this route we shall need to choose r and t_r further on in the derivation. Returning to Eq. (7), we use the fact that

$$\left[\frac{\partial}{\partial t}+\mathbf{u}\cdot\nabla-\nu\nabla^2\right]\phi = \frac{\mathbf{u}-\mathbf{u}_0}{r}\cdot\nabla\phi_0 - \frac{\nu}{r^2}\nabla^2\phi_0 \tag{8}$$

and integrate over time in the interval (t_0, t_0+t_r). We write

$$\langle w|\nabla\xi|^2\rangle_{r,t_r} = \frac{1}{\nu}(T_1+T_2+T_3+T_4)\ , \tag{9}$$

where

$$T_1 = \frac{1}{r^3 t_r}\int_{t_0}^{t_0+t_r} dt\int d\mathbf{x}\,\alpha w\phi_0\ , \tag{10}$$

$$T_2 = \frac{1}{r^3 t_r}\int_{t_0}^{t_0+t_r} dt\left[-\frac{d}{dt}\int d\mathbf{x}\, w\phi_0\right]\ , \tag{11}$$

$$T_3 = \frac{1}{r^3 t_r}\int_{t_0}^{t_0+t_r} dt\int d\mathbf{x}\left[\frac{\mathbf{u}-\mathbf{u}_0}{r}\cdot\nabla\phi_0\right] w\ , \tag{12}$$

$$T_4 = \frac{\nu}{r^5 t_r}\int_{t_0}^{t_0+t_r} dt\int d\mathbf{x}\, w\nabla^2\phi_0\ . \tag{13}$$

To proceed, denote the average

$$\langle f\rangle_{r,t} = \frac{1}{t}\int_{t_0}^{t_0+t} ds\frac{3}{4\pi r^3}\int_{|\mathbf{y}|>r} f(\mathbf{x}_0+\mathbf{u}_0 s+\mathbf{y},s)d\mathbf{y}\ . \tag{14}$$

Using this notation we can write

$$\langle w|\nabla\xi|^2\rangle_{r,t_r} \le \frac{1}{\nu}(\mathcal{T}_{\rm I}+\mathcal{T}_{\rm II}+\mathcal{T}_{\rm III}+\mathcal{T}_{\rm IV})\ , \tag{15}$$

where

$$\mathcal{T}_{\rm I} = \langle |\alpha| w\rangle_{r,t_r}\ , \tag{16}$$

$$\mathcal{T}_{\rm II} = \frac{C}{r^3 t_r}\int_{|\mathbf{y}|<r} d\mathbf{y}[w(\mathbf{y},t_0+t_r)-w(\mathbf{y},t_0)]\ , \tag{17}$$

$$\mathcal{T}_{\rm III} = \frac{C}{r}\langle |\mathbf{u}-\mathbf{u}_0| w\rangle_{r,t_r}\ , \tag{18}$$

$$\mathcal{T}_{\rm IV} = \frac{C\nu}{r^2}\langle w\rangle_{r,t_r}\ . \tag{19}$$

Here C denotes some unknown dimensionless constant, not necessarily identical in all equations where used. We have $T_1 \le \mathcal{T}_{\rm I}$ and $T_2 \le \mathcal{T}_{\rm II}$ from the boundedness of ϕ. The bound for the other two terms follows since ϕ_0 has bounded derivatives and the dependence of ϕ on $\mathbf{x}$ is through $\mathbf{x}/r$, so $\nabla\phi$ is bounded by C_1/r and $\nabla^2\phi$ by C_2/r^2, where C_1 and C_2 are constants.

At this point we use the fact that α can be written [cf. Eq. (3)] as $\alpha=\xi\cdot\mathbf{S}\cdot\xi$, where $\mathbf{S}$ is the symmetric part of $\nabla\mathbf{u}$, $\mathbf{S}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^\dagger)$. We then recognize that α is a diagonal component of $\mathbf{S}$ in the representation in which ξ is a basis vector. Therefore, we can write

$$\alpha^2 \le {\rm Tr}S^2\ . \tag{20}$$

In addition, we have the inequality

$${\rm Tr}S^2 \le |\nabla\mathbf{u}|^2\ . \tag{21}$$

We now define

$$A \equiv \langle |\nabla \mathbf{u}|^2 \rangle_{r,t_r}^{1/2} \langle w^2 \rangle_{r,t_r}^{1/2} . \tag{22}$$

With the help of the inequalities (21) and (22) and the Cauchy-Schwartz inequality we can immediately see that

$$\mathcal{T}_{\rm I} \le A . \tag{23}$$

Next we show that $\mathcal{T}_{\rm III}$ can also be bounded by A. To do so we use first the Poincaré inequality

$$\int_{|\mathbf{x}| \le r} d\mathbf{x} |\mathbf{u}-\mathbf{u}_0|^2 \le r^2 \int_{|\mathbf{x}| \le r} d\mathbf{x} |\nabla \mathbf{u}|^2 . \tag{24}$$

From the Cauchy-Schwartz principle and (25)

$$\mathcal{T}_{\rm III} \le CA \tag{25}$$

follows. The terms $\mathcal{T}_{\rm II}$ and $\mathcal{T}_{\rm IV}$ can be bounded by A by making a choice of the ball size r and the averaging time t_r. The time t_r is fixed by demanding that $\mathcal{T}_{\rm II} \le CA$. This translates to the condition

$$t_r \ge \frac{3}{4\pi r^3} \frac{\int_{|\mathbf{y}|<r} d\mathbf{y} [w(\mathbf{y},t_0+t_r)-w(\mathbf{y},t_0)]}{\langle |\nabla \mathbf{u}|^2 \rangle_{r,t_r}^{1/2} \langle w^2 \rangle_{r,t_r}^{1/2}} . \tag{26}$$

This is an implicit inequality for t_r, bounding it from below by some t_{r_0}, found by interpreting (27) as an inequality. We leave it at that for now since we do not need a precise determination of t_r for our later considerations. The ball size r, however, is very important and we obtain it by demanding that $\mathcal{T}_{\rm IV}$ is bounded by A. This translates to the condition

$$r \ge r_0 \equiv \frac{\nu^{1/2}}{\langle |\nabla \mathbf{u}|^2 \rangle_{r_0,t_r}^{1/4}} . \tag{27}$$

While these implicit definitions for r_0 and t_{r_0} look complicated, the dependence on r_0 and t_{r_0} of the averaging volume is weak and should converge very quickly. Having found r_0 we have

$$\langle w |\nabla \xi|^2 \rangle_{r_0,t_{r_0}} \le \frac{C}{\nu r_0^3 t_{r_0}} \left[\int_0^{t_{r_0}} dt \int_{|\mathbf{x}| \le r_0} d\mathbf{x} |\nabla \mathbf{u}|^2 \right]^{1/2} \times \left[\int_0^{t_{r_0}} dt \int_{|\mathbf{x}| \le r_0} d\mathbf{x}\, w^2 \right]^{1/2} . \tag{28}$$

This certainly is a finite bound: we can easily give a (far from sharp) bound if we write instead of (29) a bound that employs an integration of the whole domain L:

$$\langle w |\nabla \xi|^2 \rangle_{r_0,t_{r_0}} \le \frac{C}{\nu r_0^3 t_{r_0}} \left[\int_0^{t_{r_0}} dt \int_{|\mathbf{x}| \le L} d\mathbf{x} |\nabla \mathbf{u}|^2 \right]^{1/2} \times \left[\int_0^{t_{r_0}} dt \int_{|\mathbf{x}| \le L} d\mathbf{x}\, w^2 \right]^{1/2} . \tag{29}$$

Since $\frac{1}{2} w^2 \le |\nabla \mathbf{u}|^2$, the integral on the right-hand side is known *a priori* in terms of ϵ, the mean energy flux per unit time and mass,

$$\epsilon = \frac{\nu}{L^3} \int_{|\mathbf{x}| \le L} d\mathbf{x} |\nabla \mathbf{u}|^2 . \tag{30}$$

We can now estimate λ_c. Use (29) in (6) and that in the regions of high vorticity, and for $r \ge r_0$ the following inequality holds:

$$\langle w^2 \rangle_{r,t_r}^{1/2} \left\langle \frac{1}{w} \right\rangle_{r,t_r} \le C . \tag{31}$$

The conclusion is that

$$\lambda_c \ge C r_0 . \tag{32}$$

So our estimate for the locally averaged length scale λ_c is just the size of the smallest ball over which the length scale needs to be averaged so that we can bound the averaged λ_c. This scale can be written in a form that is more familiar to students of turbulence, i.e., that for every $r \ge r_0$

$$\lambda_c \ge \left[\frac{\nu^3}{\nu \langle |\nabla \mathbf{u}|^2 \rangle_{r,t_r}} \right]^{1/4} . \tag{33}$$

This is reminiscent of the Kolmogorov scale η,

$$\eta = \left[\frac{\nu^3}{\epsilon} \right]^{1/4} , \tag{34}$$

except that the length r_0 differs from η in that the dissipation is computed here as an average over the r ball rather than over the whole domain. Equation (33) is the central result of this section.

C. Interpretation of the results

In the preceding subsection we have derived an explicit local bound on the misalignment of the vorticity field lines. As we will see in Sec. IV, this bounds the stretching by self-induced strain. Note, however, that the bound only holds when the misalignment is averaged over a small ball, although this ball can be smaller as the local dissipation gets larger. Constantin and Fefferman show [3] that if the gradient of ξ is bounded uniformly in regions over some threshold in vorticity magnitude, a blowup is impossible and the regularity of the Navier-Stokes equations is guaranteed. In a sense this says that the only dangerous scenario for a blowup is the crossing of vortex lines. This is not protected by our bound, which involves a local average, allowing extremely short-lived events in which the vorticity can blow up.

The bound (33) is proportional to the viscosity ν. Thus for $\nu=0$ we have no bound on the divergence of direction and we cannot show that vortex lines align. This suggests that alignment is a viscosity dominated process. Scenarios have been suggested for the blowup of vorticity in the Euler equations where at least some vortex lines converge to a sharp corner [8,9]. The results shown here for viscous alignment suggest that such a blowup could be controlled by viscosity. In Sec. VI and the Appendix of this paper we analyze the effects of diffusion (as caused by viscosity) on vector fields, such as vorticity. We will see that viscous mechanisms exist that tend to align vorticity vectors.

Some effort has been made to make the bound (33) as sharp as possible. In the following we will assume that it is sharp and that we can take (33) as an equality (up to a dimensionless constant). The bound then tells us the local scale of change in field direction $|\nabla\xi|$. This is made up of a curvature component and components representing the divergence of the direction ξ in the directions orthogonal to the vortex line.

If we look at a vortex tube as an essentially coherent, aligned bundle of vortex lines, we can define a tube "width" as the diameter of that circle inside which the lines are aligned. Note that this width does not correspond to the usual definition of a vortex tube, the diameter of the semicylindrical vorticity isosurface. The radial component of $|\nabla\xi|$ represents a local measure of the typical diameter of the tube. To be precise, if we define the width of a tube as that diameter over which the direction of the field changes by some constant, the width is proportional to the average of the radial gradient of ξ over the diameter. First note that this width, the scale of the radial gradient, is larger than the scale defined by $|\nabla\xi|$ and therefore also bounded by λ_c.

If we consider the scaling behavior of the average vortex tube width (again, in the sense used here) with Reynolds number, we would tend to estimate this width as scaling like the Kolmogorov length. The averaging must exclude the low vorticity areas to ensure that (31) holds. This estimate assumes that the bound is sharp, that intermittency corrections do not induce corrections to scaling, and that no unknown physics causes the curvature to become more dominant with Reynolds number. If the local structure is not that of a vortex tube, the estimate should still hold as a misalignment scale, without the interpretation of a width. In a vorticity sheet, for example, the scale should correspond to the sheet's thickness.

To our knowledge our prediction for the typical diameters of vortex tubes, as defined by the coherence of direction, is untested in numerical experiments. In general one can estimate the scale of vortex structures by comparing terms in Eq. (1) for the vorticity; the rate of the operation of the strain over this scale should be roughly the same as the time scale of diffusion over the same scale so as to have a structure in semiequilibrium. The diffusion rate over a scale r_0 is νr_0^2. If we estimate the strain as the average straining $\sqrt{\epsilon/\nu}$, we get the Kolmogorov length as our estimate for r_0. This estimate is self-consistent since the Kolmogorov-scale strain u_η/η is also $\sqrt{\epsilon/\nu}$. One should note that if we instead estimate the strain as the integral strain U/L we get the Taylor scale. It would not, however, have been clear that such rough arguments would work for the scale of vortex alignment without our analysis of Sec. II B.

The corresponding estimate to the Taylor scale in the case of magnetohydrodynamic (MHD) flow gives the skin depth $LR_M^{-1/2}$, where R_M is the magnetic Reynolds number, which has often been proposed as the width of flux tubes in MHD flow [12]. A rigorous estimate parallel to that given here has been done [13] for the MHD case, giving as the flux tube width a new length scale that scales as $L\mathrm{Re}^{-3/4}P_M^{-1/2}$, where P_M is the magnetic Prandtl number (ratio of magnetic diffusivity and viscosity). This is also the result of the rough estimate from the magnetic field equation estimating the strain as the Kolmogorov-scale strain.

We would like to suggest here that numerical experiments can be run at a series of Reynolds numbers to test our predictions. We suggest the following way to analyze the data. First, one needs to identify the region of "higher vorticity" [to ensure that (31) holds]. This can be taken as the union of all space points in which the vorticity exceeds some fraction of its maximum. We expect that the results will be insensitive to the value of this fraction, as long as the smallest diameter of any part of this set is larger than r_0. We then need to carefully estimate λ_c itself. For this purpose $|\nabla\xi|^2$ should be computed and averaged over r_0 at each point of the higher vorticity subvolume. Using Eq. (4), λ_c can be calculated and then averaged over the entire subvolume. Moving to the next time frame the process can be repeated, until an average over time larger than t_{r_0} is obtained. Having found this estimate at one value of Re, one needs to repeat the procedure at other values of Re to compare with our expectation that $1/\lambda_c \sim O(\mathrm{Re}^{-3/4})$. Notice that in our procedure one does not need to assume that the vortex structures are circular tubes or even to identify "structures." It is a procedure that can be fully automated without ambiguities. Indeed, notice that the ball size r_0 used in the analysis and the typical scale λ_c are essentially the same. This guarantees that our construction and analysis are not going to become inappropriate or irrelevant at increasing values of Re.

One can also take Eq. (33) further to predict the variation in misalignment between points of a given simulation. The prediction is that misalignment is strongest in areas of higher dissipation. This could mean that when vortices react, a typical scenario for strong dissipation, the vortex lines are strongly distorted. This seems reasonable, although the pointwise prediction is obviously riskier than the average misalignment scale estimate. This proposal should also be examined numerically.

We can turn now to the discussion of other length scales associated with vortex tubes.

III. EULER DYNAMICS OF VORTEX TUBES: THE RADIUS OF CURVATURE

In Sec. II we focused on one length scale that characterizes the vortex tubes, i.e., the coherence length of the direction of ξ, which was interpreted as the diameter of the tubes. To see the spatial structure of the tubes we need to examine the radius of curvature and the torsion of these structures. We do this by examining the curvature and the torsion of the vortex lines, remembering that these are not quite the same thing. The radius of curvature has the usual definition $R^{-1}=|(\xi\cdot\nabla)\xi|$. We would like to examine, in this section, the dynamics of this length to show that it is indeed large in regions of high vorticity. The assumption here is that the role of viscosity is mainly to align the vortex lines into a structure with coherent dynamics, while the strain is responsible for the dynamics of this structure once formed, at least for reasonably large Reynolds numbers. We therefore restrict here our analysis to the Euler terms; the viscous effects are presented in the Appendix. We derive equations for the curvature and the torsion directly from the Euler equations. The equation we derive for the curvature can be shown to be identical to that derived by Drummond and Münch [14] (in a somewhat different way) for the curvature of material lines, not unexpectedly; we disagree somewhat, however, on the interpretation. Basically we are going to show that the same α that appears as the stretching rate of w in Eq. (2) appears again as the leading stretching term in the radius of curvature. We would like to propose that in those regions in which the vorticity grows rapidly, there is also a process of straightening up to the vortex lines.

A. Derivation of the equation of motion

We begin the derivation by considering the Frenet frame [15] of a vortex line. The unit vector in the direction of the vortex line is ξ. The second orthonormal vector, the unit vector in the direction of the curvature, is denoted by $\mathbf{n}$,

$$\mathbf{n}=\frac{(\xi\cdot\nabla)\xi}{|(\xi\cdot\nabla)\xi|} . \tag{35}$$

ξ and $\mathbf{n}$ are orthogonal since $\xi\cdot(\xi\cdot\nabla)\xi=\frac{1}{2}(\xi\cdot\nabla)\xi\cdot\xi=0$ (using $\xi\cdot\xi=1$). The third orthogonal unit vector (the binormal) will be denoted $\mathbf{b}$,

$$\mathbf{b}=\xi\times\mathbf{n} , \tag{36}$$

where the symbol $\times$ stands for a vector product. We denote the curvature by $1/R$ and the torsion by T. The Frenet equations for the variation of the binormal frame along the vortex line are

$$(\xi\cdot\nabla)\begin{bmatrix}\xi\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix}=\begin{bmatrix}0 & 1/R & 0\\ -1/R & 0 & T\\ 0 & -T & 0\end{bmatrix}\begin{bmatrix}\xi\\ \mathbf{n}\\ \mathbf{b}\end{bmatrix} . \tag{37}$$

The radius of curvature R measures the degree in which the vortex line is bent, or the deviation from a straight line. The torsion measures the twist in the direction of the curvature, or the deviation from a planar configuration. We will be interested in the dynamics of $R(x,t)$ and $T(x,t)$. We should stress that these equations are field equations for locally defined quantities and as such can be studied in numerical simulations in a completely well-defined way.

In deriving the dynamical equations of the Frenet frame we will use two facts. The first one follows from

$$[D_t,\boldsymbol{\omega}\cdot\nabla]=\mathbf{0} , \tag{38}$$

which is just the analytical expression of the fact that vortex lines are material [16] and implies that

$$[D_t,\xi\cdot\nabla]=-\alpha\xi\cdot\nabla . \tag{39}$$

The second fact is simply that if a vector $\mathbf{v}$ satisfies the equation of motion $D_t\mathbf{v}=\mathbf{F}$, then the equation for $\mathbf{v}/|\mathbf{v}|$ is

$$D_t\frac{\mathbf{v}}{|\mathbf{v}|}=P_{\mathbf{v}}^{\perp}\left[\frac{\mathbf{F}}{|\mathbf{v}|}\right] , \tag{40}$$

where $P_{\mathbf{v}}^{\perp}$ is the projection operator that projects on the plane orthogonal to $\mathbf{v}$, $P_{\mathbf{v}}^{\perp}(\mathbf{X})\equiv\mathbf{X}-(\mathbf{X}\cdot\mathbf{v})\mathbf{v}/|\mathbf{v}|^2$.

Thus we can use the Euler equation of motion of $\boldsymbol{\omega}$ itself, $D_t\boldsymbol{\omega}=\boldsymbol{\omega}\cdot\nabla\mathbf{u}$, to derive, on the one hand,

$$D_t\xi=\mathbf{S}\cdot\xi-\alpha\xi , \tag{41}$$

where $\mathbf{S}$ is the strain rate tensor, and, on the other hand,

$$D_t\xi=P_{\xi}^{\perp}\mathbf{S}\cdot\xi=(\mathbf{n}\cdot\mathbf{S}\cdot\xi)\mathbf{n}+(\mathbf{b}\cdot\mathbf{S}\cdot\xi)\mathbf{b} . \tag{42}$$

Because $\mathbf{S}$ enters these equations only through the vector $\mathbf{S}\cdot\xi$, we will denote for brevity $S_b\equiv(\mathbf{b}\cdot\mathbf{S}\cdot\xi)$, $S_n\equiv(\mathbf{n}\cdot\mathbf{S}\cdot\xi)$ and rewrite (42) as

$$D_t\xi=S_n\mathbf{n}+S_b\mathbf{b} . \tag{43}$$

Now we compute $D_t\mathbf{n}$,

$$D_t[(\xi\cdot\nabla)\xi]=(\xi\cdot\nabla)D_t\xi-\alpha(\xi\cdot\nabla)\xi . \tag{44}$$

Remembering that $|(\xi\cdot\nabla)\xi|=R^{-1}$ we get from (41)

$$D_t\mathbf{n}=RP_{\mathbf{n}}^{\perp}\left[(\xi\cdot\nabla)(S_n\mathbf{n}+S_b\mathbf{b})-\frac{\alpha}{R}\mathbf{n}\right] , \tag{45}$$

which upon using (37) simplifies to

$$D_t\mathbf{n}=-S_n\xi+R[TS_n+(\xi\cdot\nabla)S_b]\mathbf{b} . \tag{46}$$

Finally, we can write

$$D_t\mathbf{b}=D_t(\xi\times\mathbf{n})=-S_b\xi-R[TS_n+(\xi\cdot\nabla)S_b]\mathbf{n} . \tag{47}$$

Equations (43), (46), and (47) contain the full Euler dynamics of the Frenet frame. To proceed to find the dynamics of $1/R$ and T we notice that because $(\xi\cdot\nabla)\xi=R^{-1}\mathbf{n}$, we have

$$D_t\left[\frac{1}{R}\right]=\mathbf{n}\cdot D_t[(\xi\cdot\nabla)\xi] , \tag{48}$$

where (46) has been used. Writing now

$$D_t[(\xi\cdot\nabla)\xi]=(\xi\cdot\nabla)(D_t\xi)+[D_t,(\xi\cdot\nabla)]\xi , \tag{49}$$

we end up calculating

$$D_t\left[\frac{1}{R}\right]=-\frac{\alpha}{R}-TS_b+(\xi\cdot\nabla)S_n . \tag{50}$$

This equation is easily understood. A strain α that pulls the vortex on both ends straightens it out. If the normal strain in the plane of the curvature S_n increases along the vortex, the vortex will bend. If the vortex has torsion, so that it is locally wrapped helically on a cylinder, the binormal lies on the cylinder, normal to the vortex. Pulling on the coil along the cylinder with strain S_b will extend it and reduce the curvature.

In the very same way we can recognize that

$$D_tT=-\mathbf{n}\cdot D_t[(\xi\cdot\nabla)\mathbf{b}] , \tag{51}$$

which we end up calculating as

$$D_t T = -\alpha T + \frac{S_b}{R} + (\boldsymbol{\xi}\cdot\nabla)[R(\boldsymbol{\xi}\cdot\nabla)S_b + RTS_n] \ . \quad (52)$$

Equations (50) and (52) are the central results of this section.

B. Discussion of the equations of motion

The homogeneous part of Eqs. (50) and (52) takes the form

$$D_t \begin{bmatrix} R^{-1} \\ T \end{bmatrix} = \begin{bmatrix} \alpha & S_b \\ -S_b & \alpha \end{bmatrix} \begin{bmatrix} R^{-1} \\ T \end{bmatrix} . \quad (53)$$

If this were the whole equation, we could conclude that $(R^{-2}+T^2)$ decays to zero like $\exp(-2\int \alpha\, dt)$ or, in other words, the vortex tubes straighten up and become planar *at the same rate* as w increases, if α is positive.

Unfortunately, the situation is not that simple. Consider, for example, the last term in (50). Dimensional analysis would say that S_n can be of the order of magnitude of α and $(\boldsymbol{\xi}\cdot\nabla)$ can be of the order of R^{-1}. If so, the last term could cancel the first term, depriving us of the exponential growth of R.

In addition, the concept of a homogeneous term in this case becomes problematic. Drummond and Münch [14] derived an equation for the curvature of material lines in Navier-Stokes flow. Since the vortex lines are material, the two sets of equations exactly correspond, with the vorticity in our case corresponding to the line length in theirs. The equation they derive is, in our terms,

$$D_t \left[\frac{1}{R} \right] = -2\frac{(\alpha - \mathbf{n}\cdot\mathcal{S}\cdot\mathbf{n})}{R} - \sum \mathcal{W}_{ijk}\xi_i\xi_j n_k \ , \quad (54)$$

where $\mathcal{W}$ is the third rank tensor $\mathcal{W}_{ijk} = \partial^2 u_i/\partial x_j \partial x_k$. The two equations are identical since one can show that

$$\sum \mathcal{W}_{ijk}\xi_i\xi_j n_k = (\boldsymbol{\xi}\cdot\nabla)S_n - \frac{1}{R}\mathbf{n}\cdot\mathcal{S}\cdot\mathbf{n} + \frac{1}{R}\alpha - TS_b \ . \quad (55)$$

Drummond and Münch too assume that their homogeneous term, different from ours, dominates, at least for short times. In fact, in the two-dimensional case they use this [17] to deduce that the pth log-cumulant moment of the curvature goes as $(-3)^p$ times the pth log-cumulant moment of the line length (parallel to our w). They then verify this using, for the velocity field, a random Gaussian field.

To understand the difference, notice that in order to have an appreciable effect the strain must operate on the vorticity field over some length of time in a coherent fashion. In order for the other strain terms to have a countereffect on the vortices they must act coherently through this time with α. If we assume that in general the strain fluctuates rapidly in the turbulent field, the behavior will be dominated by the time-coherent terms.

In our case during the vortex growth the stretching rate α must behave coherently in time in order to create appreciable amplification of vorticity, since we are looking precisely at strong amplitude vortices. The question is what other terms act coherently with α. For example, in the two-dimensional case, by incompressibility $\mathbf{n}\cdot\mathcal{S}\cdot\mathbf{n} = -\alpha$. In the random Gaussian field that Drummond and Münch use for their velocity it is reasonable to expect that the $\nabla\mathbf{u}$ tensor is decorrelated from the $\nabla\nabla\mathbf{u}$ tensor. Then one can expect indeed that the curvature grows with $\exp(-3\alpha t) \sim w^{-3}$ as Drummond and Münch predict and verify.

However, in the general three-dimensional case it is hard to predict what correlations the structure of the flow will create and how relevant they will make relations such as (55). The issue can probably be settled only by an analysis using numerical simulations. It seems to us that our way of writing the equation, with the orthogonal components of the strain separated, each acting independently on the geometric properties of the field, is likely to be a good guide to understanding the dynamics.

In summary, although we cannot prove that the radius of curvature grows at the same rate of the vorticity magnitude when the latter grows, we have given arguments in favor of such a possibility. The picture that emerges is of vortex tubes in which the high vorticity values concentrate and which have a diameter of λ_c and a radius of curvature that tends to grow at the same rate of the vorticity. At any rate we certainly would predict that the stronger the tubes are, the straighter they are, since while the strain component α both straightens and amplifies, the other strain components have no obvious connection to the tube amplitude. We think that this picture can and should be tested in numerical simulations and we shall return to this issue later.

IV. STRETCHING RATE OF THE VORTICITY AND REGULARITY

A. Preliminaries

In this section we analyze the rate of stretching $\alpha = \boldsymbol{\omega}\cdot(\boldsymbol{\omega}\cdot\nabla\mathbf{u})/w^2$. If this quantity could be bounded from above, we could prove that the initial value problem does not generate spontaneously finite time singularities. We shall not be able to bound this quantity. Rather we shall explore the implications of the picture obtained above in which the regions of high vorticity seem to form elongated tubes of diameter λ_c, a diameter that is bounded from below, and of radius of curvature R that tends to grow exponentially with the same rate α. We shall show that this picture has an interesting bearing on the analysis of α, with a possibility of a link between the mechanism explored here and the possible regularity of the Navier-Stokes equations.

We analyze the rate of stretching of the vorticity in a very simple situation. We will make the assumption, in line with our remarks in Sec. II C, that almost all the variation in the vortex lines in the strong tubes comes from the curvature of the tubes. More precisely, in the regions of high vorticity we will make the assumption that the alignment of the vortex lines is so strong that the component of the gradient of ξ orthogonal to ξ is much smaller than the curvature. In other words, we will assume that for any two points $\mathbf{x}$ and $\mathbf{x}+\mathbf{y}$ well within the intense vortex tubes,

$$|\xi(\mathbf{x}+\mathbf{y})-\xi(\mathbf{x})| \le \frac{|y|}{R} \tag{56}$$

for all $|\mathbf{y}| \le \lambda_0$. The image behind this assumption is the one shown in Fig. 1. The intense vorticity regions that have a radius λ_0 are made of locally parallel vortex lines, which curve on the scale of R. Outside a λ_0 neighborhood w is small and the direction ξ becomes quite random. We would assume that λ_0 is proportional to the λ_c we found in Sec. II since we assume that this is the length scale characterizing the tube diameters. However, this assumption is not really critical for our analysis: what we need is that a well-aligned core with diameter λ_0 exists where curvature is dominant.

B. Stretching rate

With these preliminaries we can turn now to the calculation of the stretching rate $\alpha(\mathbf{x})$. We first recall that the stretching rate α can be thought of as a diagonal element of $\mathbf{S}$ in a representation in which ξ is one of the coordinate vectors, i.e.,

$$\alpha(\mathbf{x})=\xi\cdot[\mathbf{S}(\mathbf{x})\cdot\xi] \ . \tag{57}$$

We can write this in a more useful way by using the Biot-Savart law that relates the vorticity to the velocity field [3]:

$$\mathbf{u}(\mathbf{x})=-\frac{1}{4\pi}\int\left[\nabla\frac{1}{|\mathbf{y}|}\right]\times\boldsymbol{\omega}(\mathbf{x}+\mathbf{y})d\mathbf{y} \ . \tag{58}$$

From this relation one can calculate [3] the strain tensor $\mathbf{S}(\mathbf{x})$ and the stretching rate $\alpha(\mathbf{x})$, respectively,

$$\mathbf{S}(\mathbf{x})=\frac{3}{4\pi}\int\frac{1}{2}[\hat{\mathbf{y}}(\hat{\mathbf{y}}\times\boldsymbol{\omega})+(\hat{\mathbf{y}}\times\boldsymbol{\omega})\hat{\mathbf{y}}]\frac{d\mathbf{y}}{|\mathbf{y}|^3} \ , \tag{59}$$

$$\alpha(\mathbf{x})=\frac{3}{4\pi}\int\frac{1}{2}(\hat{\mathbf{y}}\cdot\xi)\det[\hat{\mathbf{y}},\boldsymbol{\omega}(\mathbf{x}+\mathbf{y}),\xi(\mathbf{x})]\frac{d\mathbf{y}}{|\mathbf{y}|^3} \ . \tag{60}$$

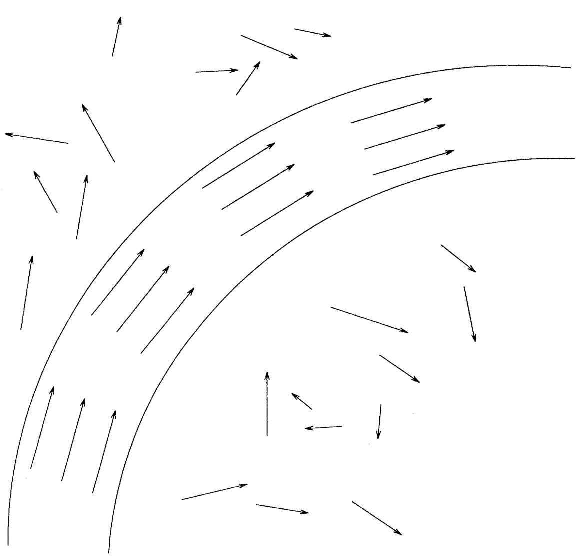

FIG. 1. Well-aligned vortex tube. All vortex lines in the tube are parallel. Outside the tube the alignment is weak.

In (59) and (60) $\hat{\mathbf{y}}$ is a unit vector in the local direction of $\mathbf{y}$ and $\det(\mathbf{a},\mathbf{b},\mathbf{c})$ is the determinant of the matrix whose columns are the three column vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. The analysis of $\alpha(\mathbf{x})$ when $\mathbf{x}$ belongs to a region of low vorticity is relatively easy and is not pursued here. The important regions for estimating α are of course the intense vortex tubes, when $\mathbf{x}$ is in a λ_0 neighborhood of intense vorticity. To analyze it we separate the contributions to the integral (60) to those coming from points $\mathbf{x}+\mathbf{y}$ which are in a λ_0 neighborhood and to the rest of the world:

$$\alpha(\mathbf{x})=\alpha_{\rm in}(\mathbf{x})+\alpha_{\rm out}(\mathbf{x}) \ , \tag{61}$$

$$\alpha_{\rm in}(\mathbf{x})=\frac{3}{4\pi}\int_{|\mathbf{y}|\le\lambda_0}\frac{1}{2}(\hat{\mathbf{y}}\cdot\xi)\det[\hat{\mathbf{y}},\boldsymbol{\omega}(\mathbf{x}+\mathbf{y}),\xi(\mathbf{x})]\frac{d\mathbf{y}}{|\mathbf{y}|^3} \ , \tag{62}$$

$$\alpha_{\rm out}(\mathbf{x})=\frac{3}{4\pi}\int_{\lambda_0\le|\mathbf{y}|\le L}\frac{1}{2}(\hat{\mathbf{y}}\cdot\xi)\det[\hat{\mathbf{y}},\boldsymbol{\omega}(\mathbf{x}+\mathbf{y}),\xi(\mathbf{x})]\frac{d\mathbf{y}}{|\mathbf{y}|^3} \ . \tag{63}$$

On the face of it, Eq. (62) seems dangerous since it can exhibit a singularity at the lower bound of the integral. It is here that Eq. (56) becomes crucial. With its help, Eq. (62) is estimated as follows: denoting $Q\equiv(\hat{\mathbf{y}}\cdot\xi)\det[\hat{\mathbf{y}},\boldsymbol{\omega}(\mathbf{x}+\mathbf{y}),\xi(\mathbf{x})]$ we notice that only the projection of $\boldsymbol{\omega}(\mathbf{x}+\mathbf{y})$ that is orthogonal to $\xi(\mathbf{x})$ contributes to Q. Denoting the angle between $\xi(\mathbf{x})$ and $\xi(\mathbf{x}+\mathbf{y})$ as ϕ, i.e.,

$$\cos(\phi)=\xi(\mathbf{x})\cdot\xi(\mathbf{x}+\mathbf{y}) \ , \tag{64}$$

we can estimate

$$Q\le w(\mathbf{x}+\mathbf{y})|\sin(\phi)| \ . \tag{65}$$

Using the fact that $|\sin(\phi)|\le|2\sin(\phi/2)|$ we conclude (see Fig. 2) that

$$Q\le w(\mathbf{x}+\mathbf{y})|\xi(\mathbf{x}+\mathbf{y})-\xi(\mathbf{x})| \ . \tag{66}$$

Using Eqs. (66) and (56) in Eq. (62) we find that

$$|\alpha_{\rm in}(\mathbf{x})|\le\frac{1}{R}\int_{|\mathbf{y}|\le\lambda_0}\frac{d\mathbf{y}}{|\mathbf{y}|^2}w(\mathbf{x}+\mathbf{y})\le w_{\max}\frac{\lambda_0}{R} \ , \tag{67}$$

where

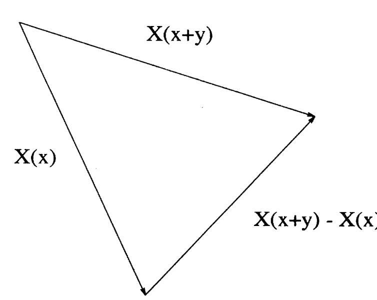

FIG. 2. Typical orientation of the unit vectors $\xi(\mathbf{x})$ and $\xi(\mathbf{x}+\mathbf{y})$ and the difference between them $\xi(\mathbf{x})-\xi(\mathbf{x}+\mathbf{y})$.

$$w_{\max} = \sup_{|\mathbf{y}| \le \lambda_0} [w(\mathbf{x}+\mathbf{y})] \ . \tag{68}$$

This is the central result of this section and it will be discussed in detail in the following subsection. It means that since the vorticity elements stretching another vorticity element must be at an angle to it, the rate of stretching is proportional to the gradient in direction, or to $1/R$.

The outer contribution (63) is not dangerous. It is estimated to be

$$\begin{aligned} |\alpha_{\text{out}}(\mathbf{x})| &\le \int_{\lambda_0 \le |\mathbf{y}| \le L} \frac{d\mathbf{y}}{|\mathbf{y}|^3} w(\mathbf{x}+\mathbf{y}) \\ &\le \left[\int_{0 \le |\mathbf{y}| \le L} w^2 d\mathbf{y} \right]^{1/2} \left[\int_{\lambda_0 \le |\mathbf{y}| \le L} \frac{d\mathbf{y}}{|\mathbf{y}|^6} \right]^{1/2} \end{aligned} \tag{69}$$

The first root on the right-hand side is just the L^2 norm and we end up estimating

$$|\alpha_{\text{out}}(\mathbf{x})| \le \|w\|_{L^2} \left[\frac{\lambda_0}{L} \right]^{-3/2} \tag{70}$$

C. Discussion of the results (67) and (70)

The most elementary scenario for the blowup of vorticity is that in which a curved vortex induces a strain that in turn stretches the vortex, creating a feedback mechanism giving a quadratic growth rate of the vorticity and thus a finite time blowup. This scenario is also very dangerous since unlike hairpin blowups, which require quite specific initial conditions, the requirements for this picture are very mild. All that is needed is a vortex that stays curved for a period of time. The results of the preceding subsection give a plausible mechanism for evading such a blowup, where as the vorticity inducing the growth rate grows, the geometry straightens out, and the ability to induce strain decreases. We have indicated that we think it likely that the dominant rate of growth of the radius of curvature during vortex growth is equal to that of the vorticity. If we accept the picture of Drummond and Münch, in the case of strong α the rate of straightening is even faster than the rate of vorticity growth. Note, however, that other components of strain besides α can act to keep the curvature strong at least for part of the development. In fact, this implies that it is possible that for the very strongest tubes the correlation between strength and straightness is attenuated somewhat since those tubes that manage to stay curved due to nonhomogeneous terms for some time will grow faster due to the strain self-induced by this curvature.

The mathematical theory of the Navier-Stokes has so far not provided a proof of the regularity of solutions. Here we have taken some steps towards a more physically oriented analysis of this question. The importance is in the possibility of understanding the nature and origin of strong events occurring in turbulent flow in nature.

V. IMPLICATIONS OF THE SCALING THEORY OF TURBULENCE

In this section we make the connection to the often discussed possibility of a deviation from the Kolmogorov scaling picture and the phenomenon of intermittency. In another paper [5] two of the present authors examined the scaling theory of turbulence with the help of the metric properties of the graphs of the velocity field in $3+3$ dimensions [4]. The main conclusion there was that in order to have deviations from the classical scaling picture one needs to see a tendency towards the creation of two-dimensional structures. By "two-dimensional" structures we mean a flow geometry in which the strain tensor $\mathbf{S}$ has a zero eigenvalue with the vorticity being the appropriate eigenvector (i.e., $\mathbf{S}\cdot\boldsymbol{\omega}=0$). An isolated straight vortex tube, in which the strain in the plane is orthogonal to the tube, is a simple example of such a structure. We have argued that if the mean eigenvalues of $\mathbf{S}$ are denoted $\langle\lambda_1\rangle \ge \langle\lambda_2\rangle \ge \langle\lambda_3\rangle$ (knowing that $\lambda_1+\lambda_2+\lambda_3=0$ in incompressible flows), then deviations from the Kolmogorov picture are expected only if the ratio $\langle\lambda_2\rangle/\langle\lambda_1\rangle$ has a Re number dependence. Explicitly, we argued that if this ratio of mean eigenvalues depends on Re like $\text{Re}^{-\beta}$, then the scaling exponent of velocity structure functions can become as high as $(1+\beta)/(3-\beta)$, compared with the classical prediction of $\frac{1}{3}$. The Re dependence of the ratio $\langle\lambda_2\rangle/\langle\lambda_1\rangle$ can be deduced from the numerical simulations in [2]. In order to have such a Re dependence we need to understand why as Re increases there is more and more tendency to create locally two-dimensional structures.

We believe that the present study is a step in the direction of understanding this phenomenon. We have seen above that the vorticity tends to concentrate in elongated tubelike structures. If all the intense vorticity regions have a large radius of curvature and the dominant strain in these regions is induced by the local vorticity, these intense strains will tend to align orthogonally to the tubes. At least locally the vorticity becomes an eigenvector of $\mathbf{S}$ with eigenvalue 0. If this phenomenon becomes more important at high Re numbers, we can expect that the ratio $\langle\lambda_2\rangle/\langle\lambda_1\rangle$ would decrease when Re increases, as is indicated by some numerical simulations [2].

What is sorely missing in our picture is a handle on the statistics. We imply that the Kolmogorov statistics should apply in the region of typical vorticity magnitude, far from the tubes of intense vorticities. The interesting dynamics and all the different physics is in the vortex tubes, but we cannot at this stage assess what is the contribution of each region to the global averages. In fact, if we accept the notion that the high vorticity regions concentrate on fractal regions [18] we would expect that their contributions to the averages of velocity differences (structure functions) would become smaller for large values of Re. It is possible that after some intermediate asymptotics in which $\langle\lambda_2\rangle/\langle\lambda_1\rangle$ decreases with Re we would reach a limit in which $\langle\lambda_2\rangle/\langle\lambda_1\rangle\rightarrow$const and asymptotically the Kolmogorov exponents for the structure functions would be resurrected. For more evidence for this view, see [19]. Obviously the question of statistical averages is wide open. This crucial question, whose solution is needed in order to offer a reasonable connection to the scaling theory, must be left for future study, presumably with the help of numerical simulations.

In numerical simulations one can address the question of conditional contributions to the overall averages, depending on the value of the vorticity. The eigenvalues of the strain tensor can be sampled conditionally and the contribution to any Re dependence can be ascribed to the vortex tubes if it really comes from there. We think that such numerical experiments can shed very important light on the dynamics responsible for the scaling theory of turbulence.

VI. EFFECT OF DIFFUSION ON VECTORS: THE SIMPLEST CASE

In this section we will examine the effect of diffusion on vector fields as a step towards understanding the effect of viscosity on vorticity structures. The question is one of some generality. In contrast to the case of the diffusion of scalar fields, the interest comes from the fact that we are looking at the behavior not of the vector components, which indeed diffuse simply as scalars, but at rather odd functions of them, namely, the amplitude and the direction of the vectors—the quantities of interest when looking at structures. In addition we are looking not at asymptotic but at short time behavior, relevant for coherent structures with finite lifetime. The object is to see how viscous mechanisms could lead to alignment of vortex lines in tubes. To do this we first examine the simplest dimensional case, that of two-dimensional vectors on the line.

In this case we find that if the vector field is written as

$$\mathbf{v}=A(\sin(\theta),\cos(\theta)) , \tag{71}$$

then

$$\Delta\mathbf{v}=\left[\frac{1}{A}\frac{d^2A}{dx^2}-\left[\frac{d\theta}{dx}\right]^2\right]\mathbf{v}+\left[A\frac{d^2\theta}{dx^2}+2\left[\frac{d\theta}{dx}\right]\left[\frac{dA}{dx}\right]\right](\cos(\theta),-\sin(\theta)) \tag{72}$$

or, purely in terms of amplitude and angle,

$$\frac{dA}{dt}=\Delta A-|\nabla\theta|^2A , \tag{73}$$

$$\frac{d\theta}{dt}=\Delta\theta+\frac{2}{A}\nabla\theta\cdot\nabla A . \tag{74}$$

(These are just Kida and Takaoka's [20] P and R as we define them in the Appendix.) One expects the amplitude to decay quickly (exponentially) in areas of variation in θ due to the term quadratic in the variation. This already shows that nonaligned regions will tend to be of weak amplitude. In addition the amplitude is dissipated as a scalar.

The equation for the angle shows that the angle changes more slowly, being diffused by the first term. The second mixed term represents an interaction between variation in angle and amplitude. Let us see the effect of these two terms.

We first consider the angle diffusion term. Consider a linear array of vectors of constant magnitude, with a constant angle difference between vectors. The above implies that the vector directions will not change in time. As with scalar diffusion, there is not diffusion of a constant gradient, as determined by the border conditions. The magnitude of all the vectors decays at the same rate. The same holds true for a circle of vectors of constant magnitude all pointing in the radial direction. (This last case will appear in our analysis of three-dimensional vortices in the Appendix.)

To understand nonstatic situations we assume first that we have only the angle diffusion term: for short times, as in scalar diffusion, a localized sharp change in angle or angle gradient will be smoothed in time over a larger and larger range. (In both these cases the vectors in the vicinity of the kink will shrink strongly with time due to the second term in the equation for the amplitude.) An angle gradient with a cubic dependence on space will create an increasing gradient in the direction of the cubic correction: $y=\pm\alpha x^3\pm 6\alpha t x$ is a solution to the one-dimensional (1D) diffusion equation, in which the gradient grows in principle without limit. For example, if a gradient starts to splay outwards in a cubic fashion on the edges, the gradient in the center will spread out to accommodate it.

We now turn to the effect of the second (coupling) term. First we look at the situation of a constant amplitude gradient

$$A(x)=A_0+\alpha x . \tag{75}$$

As a scalar, this would not be diffused. Given a direction gradient, the angle will be revolved, with the rate increasing with the amplitude:

$$\frac{d\theta}{dt}=\frac{2\alpha}{A_0+\alpha x}\nabla\theta . \tag{76}$$

All vectors are revolved, but the larger they are, the less they revolve. Thus the diffusion tends to align the vectors with the largest vector. Near a simple maximum in amplitude

$$A(x)\sim A_0-\alpha x^2 \tag{77}$$

with a linear angle gradient we can solve exactly: writing the vector as a complex number and approximating the amplitude as an exponent, the initial vector is

$$V(t=0)=e^{-\alpha x^2+ikx} . \tag{78}$$

We find the solution of the diffusion equation by completing the square as

$$V(t)=\exp\left[\frac{-\alpha x^2}{\sqrt{t+1}}+\frac{ikx}{\sqrt{t+1}}-\frac{k^2t}{8\alpha^2}\right] . \tag{79}$$

The phase gradient $k/\sqrt{t+1}$, in contrast to the case of pure angle diffusion, *decreases* in time, increasing alignment. In this "near maximum" approximation the alignment is uniform and to get real bunching (i.e., an increased alignment near the maximum) we need a generic higher term, say a dependence on x^3. On the other hand, as we have seen, the effect of the first term alone on such a gradient is the opposite—to spread it out instead of bunching it. A profile can be found, in fact, for which these two effects exactly balance each other: near a maximum $A \approx 1-\alpha x^2$ the two terms balance for an angle gradient of $\theta \approx x + \frac{2}{3}\alpha x^3$. A gradient steeper in angle will be spread by diffusion and a more gradual one will be bunched around the largest vector. In the same way near a minimum the vectors will be repulsed from the smallest vector. These are only the initial dynamics of simple situations and further development would evolve the amplitude to a less simple configuration. The conclusion is that in general diffusion has a tendency to align in the direction of the largest vector.

In the full dynamics, including strain, we expect these ideas to still hold. If some local strain creates a sharp enough peak in vorticity strength, the diffusion should align the vectors in this region with the large vorticity vectors.

We tested out these ideas by numerical simulations. Pure diffusion is trivial to integrate using the Fourier transform (the integration is done, of course, not in the amplitude and angle variables but in the usual vector components.) Simulations starting from an amplitude with strong peaks indeed show the phenomena of aligning around maxima and repulsion from minima (see Fig. 3). (Periodic functions for initial amplitude and angle were used in order to allow an exact solution.) We see that near the peaks in amplitude the angle gradient is suppressed.

A much greater effect was achieved by combining strain and diffusion. Strain was added by integrating for the two vector components the equation

$$\frac{dv_i}{dt} = S(x)v_i + \nu \Delta v_i \tag{80}$$

using Interactive Mathematics Software Library/Interactive Data Language Adams-Gear ordinary differential equation routine. This isotropic strain (diagonal with equal elements) alone should not realign the vector directions. Indeed when a peaked strain field was used, a comparison with the same integration without diffusion, or to integration without strain, shows the aligning and repulsion effects clearly (see Figs. 4–6).

To summarize we can say that the alignment is greater in strong areas, on the one hand, because the amplitude is reduced by direction misalignment. On the other hand, strong amplitudes increase alignment by dynamic aligning of neighboring vectors. In areas of weak amplitude

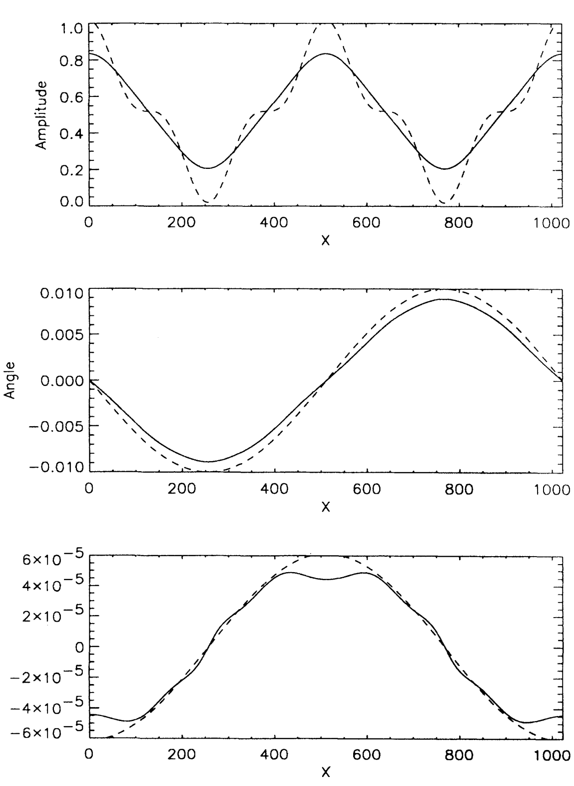

FIG. 3. Simulation of diffusion of two-vectors on a line. The solid line is after integration and the dashed line is the initial function. (a) The vector amplitude, (b) the vector angle, and (c) the gradient of the vector angle. Note the alignment around the amplitude peak. All units are arbitrary.

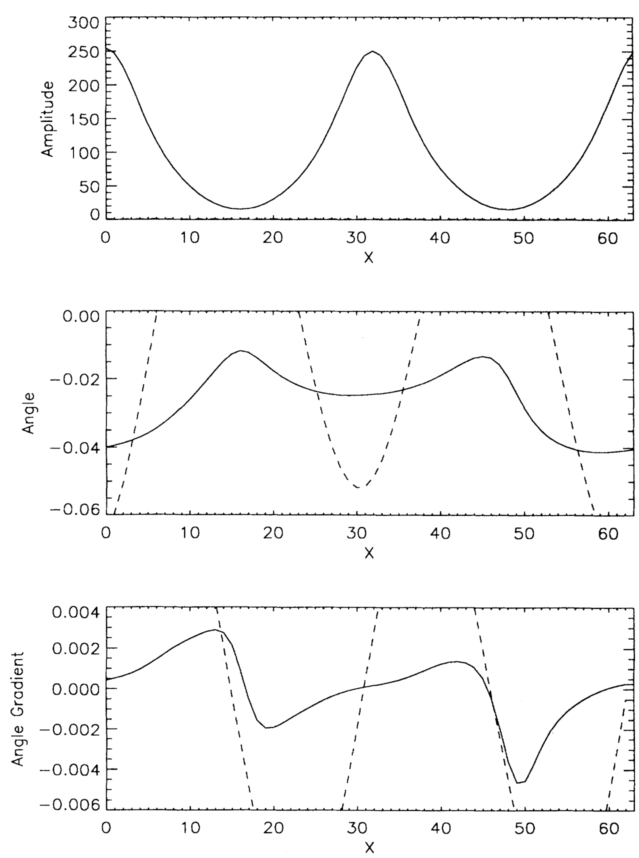

FIG. 4. Simulation of diffusion and strain of two-vectors on a line. The solid line is after integration and the dashed line is the initial function. (a)–(c) are the same as in Fig. 3.

diffusion tends to spread out angle gradients.

The vector line of Eq. (71) will describe, when we address the three-dimensional problem, the dynamics of the vectors on the plane tangent to the vector line. To see what the dynamics are along the vortex line, we will look at the case of what we will call *diffusing filaments.* These are constructed in the same way as vector lines in a vector field: we start as in Eq. (71) with a line of 2D vectors. We then construct a line in the 2D plane so that the direction of the tangent to the filament is given by the direction of the vector at each point. The magnitude on the line is given by the vector amplitude. The unit vector in the tangent direction at each point is ξ. In these terms

$$|\nabla\theta| = |(\xi\cdot\nabla)\xi| = c \ , \tag{81}$$

c being the curvature of the filament.

We now assume that these vectors obey a 1D diffusive dynamics along the filament. The equations are just the same as in Sec. V, but in our new terms translate to

$$\frac{dw}{dt} = \Delta w - c^2 w \ , \tag{82}$$

$$\frac{d\xi}{dt} = (\xi\cdot\nabla)c + \frac{2}{w}\xi\cdot\nabla w \ . \tag{83}$$

Thus the amplitude diffuses along the filament, with an

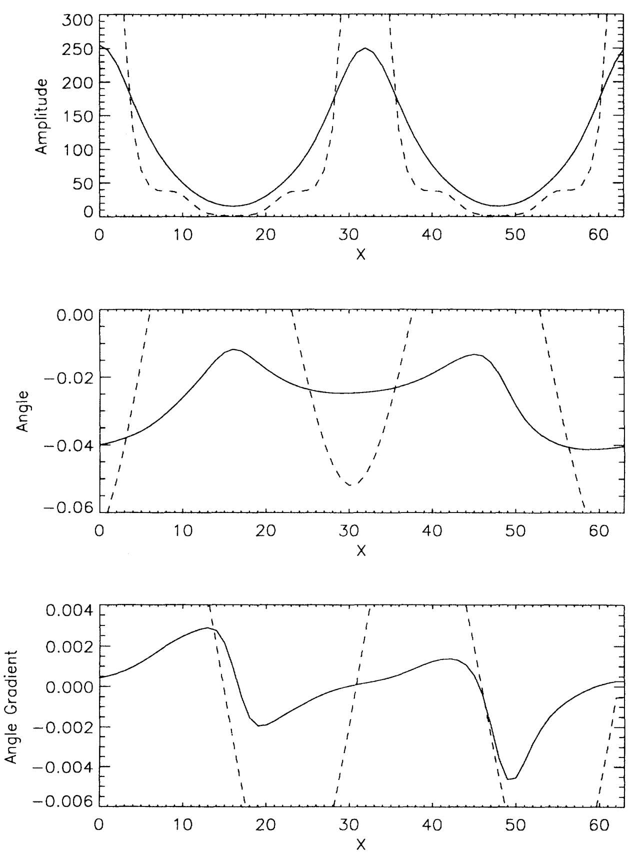

FIG. 5. Simulation of diffusion and strain of two-vectors on a line. The solid line is after integration and the dashed line is the integration of strain only. (a)–(c) are the same as in Fig. 3.

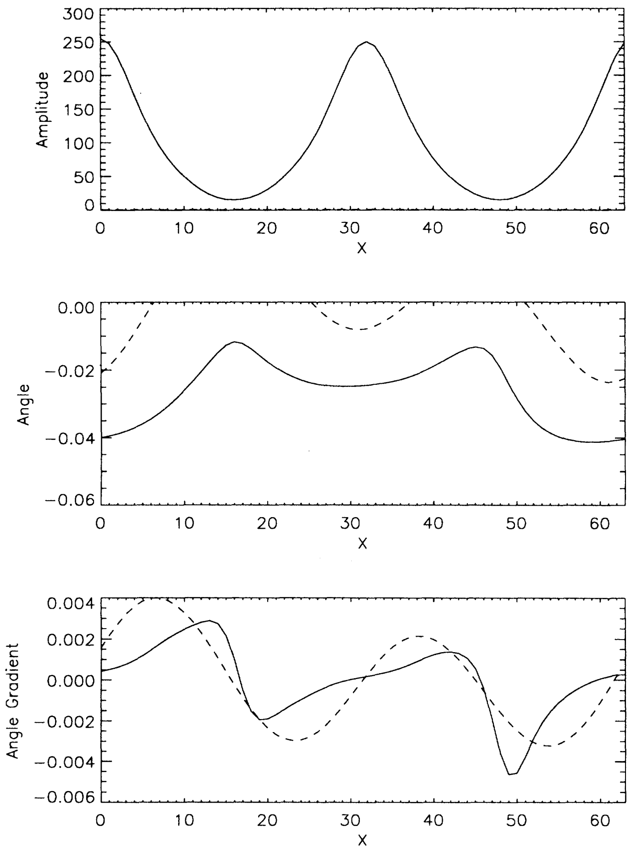

FIG. 6. Simulation of diffusion and strain of two-vectors on a line. The solid line is after integration and the dashed line is the integration of diffusion only. (a)–(c) are the same as in Fig. 3.

added exponential decay from curvature. The dynamics of ξ show two different terms. The first, corresponding to the diffusion of θ, acts to diffuse away maxima like any diffusion. Since a maximum angle exists at points of inflection, the diffusion will flatten these and thus tend to straighten the curve. The second terms tends to align the tails of a filament segment of strong magnitude with that segment, if they are curved. Thus diffusion on a filament tend to straighten, to diffuse amplitude, and to reduce curvature. We can see in Fig. 7 how filaments are "ironed out" by diffusion. Such filaments can be drawn and their diffusion integrated in three dimensions with very similar results.

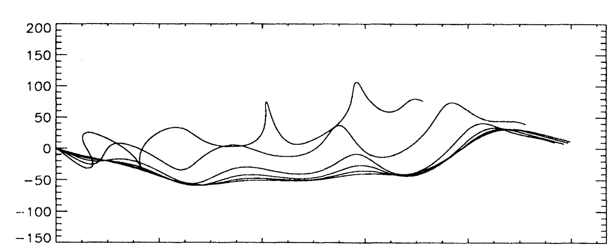

FIG. 7. Diffusion of vector filaments.

The full three-dimensional case of diffusion of vectors, in particular the question of alignment of vortex tubes, is much more complex. In the Appendix we present the equations governing the vortex lines and will see that they may well be largely understandable in terms of the simple cases considered here.

VII. CONCLUSIONS

We have presented a selection of results on the structure of the field of vortex lines in turbulent flow. All the results are connected in some way with the question of the blowup of vorticity in Navier-Stokes flow since it is the geometry of the vortex lines that determines how the vorticity induces that strain that in turn amplifies the vorticity. We think that the results are also interesting in their own right.

We first bound the variation in vorticity direction. This bound is ascribed to viscous effects. The result controls the possible magnitude of the locally induced strain. If indeed blowups do not occur this could well be an important effect. The bound is also interpreted as an estimate for the typical width of vortex tubes as observed in simulations.

We analyze the nonviscous dynamics of the curvature and torsion of the vortex lines. We show that strong vortex tubes should be straighter and that it is likely that these tubes will straighten out as they are amplified (a stronger assumption).

We then examine the stretching rate of the vorticity, in particular its self-stretching (that locally induced). We see that the curvature dynamics implies that self-stretching will be limited, and finally controlled, by the straightening of the vortices.

The straightening of the vortices is also linked to possible departures from simple (Kolmogorov) scaling. The resultant local tendency to two dimensionality leads to strain anisotropy, which has been shown to be a condition for a departure from scaling.

The effect of viscosity on the vorticity vectors is analyzed in Sec. VI and in the Appendix. Among other things we see that the viscosity can indeed enhance alignment locally due to effects of dynamic alignment of vortex lines. It remains to check in numerical simulations whether the time scales of these effects allow them to make a significant difference to the field vector alignment.

The dependence of vorticity alignment on the strength of the local vorticity is seen to be complex. On the one hand, dynamic alignment should be enhanced in regions of strong vorticity. The viscous effects shown here tend to work better in regions of strong vorticity, while a coherent strain would tend to create by stretching a region of vorticity vectors coherent in direction too. On the other hand, vortex line divergence can create locally induced strain leading to nonlinear growth, thus causing a correlation of magnitude to misalignment. This means that while it is difficult to predict the dependence of alignment on vorticity strength, the results contained here should allow a more ready understanding of numeric measurements of such a dependence.

ACKNOWLEDGMENTS

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APPENDIX: VISCOSITY AND CONVECTION IN VORTICES: THE THREE-DIMENSIONAL CASE

We will speak here occasionally of the fate or evolution of vortex lines, although these are not defined in the full sense in the viscous case, not being material. They are defined at any given moment for a given vector field away from zeros of the field and their configuration has to change smoothly. We will speak about the behavior of lines only in such situations and in this sense as an aid to understanding. We will speak of vortex tubes in the same sense—as far as in some (possibly Lagrangian) region the vortex lines from moment to moment tend to stay in a fairly coherent, fairly aligned bundle, we will refer to this as a vortex tube.

The viscous term in the vorticity equation was decomposed by Kida and Takaoka [20] in the following way: the viscous term $\nu\Delta\boldsymbol{\omega}$ is divided into the component in the direction of the vorticity,

$$P=\boldsymbol{\xi}\cdot\Delta\boldsymbol{\omega} \tag{A1}$$

and the orthogonal component $\mathbf{R}=\Delta\boldsymbol{\omega}-\boldsymbol{\xi}P$. The former expresses the difference between the stretching of the material element and that of the vorticity and therefore the dissipation (or growth) of vorticity magnitude; the latter expresses the rotation of the vorticity vector off the material line, leading to displacement of the vortex off the material line. In fact, Kida and Takaoka originally proposed the latter quantity as a probe or marker for reconnection activity in simulation data. This decomposition of the viscous force corresponds to considering the magnitude w and direction of the vorticity separately; doing this we see that the viscous terms for the vorticity evolution are

$$D_t w=\nu P\ , \tag{A2}$$

$$D_t\boldsymbol{\xi}=\frac{\nu}{w}\mathbf{R}\ . \tag{A3}$$

Both can act to bring about a vortex field exhibiting alignment: the P by killing the field where not aligned, the $\mathbf{R}$ by rotating the vectors so that they point in the same direction.

By substituting $\boldsymbol{\omega}=w\boldsymbol{\xi}$ we find that

$$\Delta\boldsymbol{\omega}=\boldsymbol{\xi}\Delta w+w\Delta\boldsymbol{\xi}+2(\nabla w\cdot\nabla)\boldsymbol{\xi} \tag{A4}$$

so

$$P=\Delta w+w\boldsymbol{\xi}\cdot\Delta\boldsymbol{\xi}+2\boldsymbol{\xi}\cdot(\nabla w\cdot\nabla)\boldsymbol{\xi}\ . \tag{A5}$$

The first term expresses the expected simple diffusion of

w, which acts to smooth out the w distribution, and acts against the strain creating and stretching the vortex. The second term is always negative; in fact

$$\xi\cdot\Delta\xi=-|\nabla\xi|^2 \ . \tag{A6}$$

Thus any spacial variation in ξ causes a drain in the vorticity.

We will now represent everything in the binormal frame. This will allow us to understand the equations in terms of the simpler dynamics of Sec. VI. In terms of the binormal basis $(\xi,\mathbf{n},\mathbf{b})$ the second term is

$$|\nabla\xi|^2=c^2+|(\mathbf{n}\cdot\nabla)\xi|^2+|(\mathbf{b}\cdot\nabla)\xi|^2 \ , \tag{A7}$$

where c is the curvature, so we have a contribution from curvature and from orthogonal gradients of ξ, i.e., divergence of lines in the orthogonal plane. The third term when expressed using the binormal basis is

$$\xi\cdot(\nabla w\cdot\nabla)\xi=[(\mathbf{n}\cdot\nabla)w]\xi\cdot(\mathbf{n}\cdot\nabla)\xi+[(\mathbf{b}\cdot\nabla)w]\xi\cdot(\mathbf{b}\cdot\nabla)\xi \tag{A8}$$

and is therefore zero. So

$$P=\Delta w-w|\nabla\xi|^2 \ . \tag{A9}$$

Thus with respect to P our analysis of the full three-dimensional case just corresponds to our analysis of the simple 2D case. The same is not true, however, for R where other types of viscous effects appear. The component of R in the n direction is

$$R_n=w\mathbf{n}\cdot\Delta\xi+2[(\mathbf{n}\cdot\nabla)w]\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\xi \\ +2[(\mathbf{b}\cdot\nabla)w]\mathbf{n}\cdot(\mathbf{b}\cdot\nabla)\xi+2c[(\xi\cdot\nabla)w] \ . \tag{A10}$$

The τ component can be found similarly and has the same form except that it is lacking the last term.

Let us examine some more special cases to see what effect these terms would have. There are two opposing types of vortices we can look at (see Fig. 8). One is a vortex where all the ξ are perfectly aligned and vary only in the direction of ξ itself due to curvature and torsion; this we call a *cylindrical* vortex. (This configuration is not realizable globally, but we will assume that that is handled through a long-range cutoff function that does not make much difference to the derivatives near the vortex center.) These are reminiscent of the filaments of Sec. VI, but exist in three dimensions. The second example is that of an axially symmetric vortex with a straight backbone, possibly with cross section undulating along its length, which we call a *neck* vortex. This is the important example for the study of alignment since it shows us whether viscosity keeps vortices aligned.

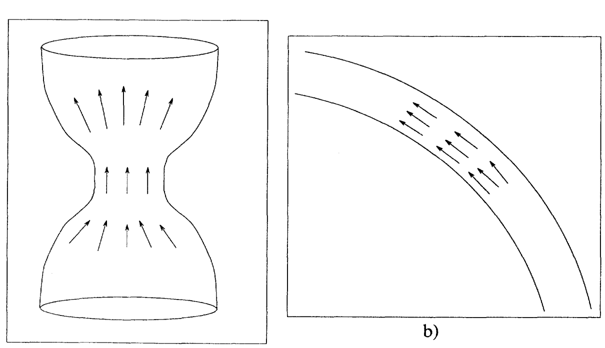

FIG. 8. Two types of simplified vortices: (a) cylinder vortices and (b) neck vortices.

The simplest 3D effect of viscosity on a vortex is *dissipation of circulation*,

$$D_t\Gamma=D_t\int\boldsymbol{\omega}\cdot d\mathbf{a}=\nu\int\Delta\boldsymbol{\omega}\cdot d\mathbf{a} \ , \tag{A11}$$

the integration being over some area. We choose the area of a circle centered on the vortex backbone and orthogonal to the vortex direction there. In the case of the cylindrical vortex $d\mathbf{a}$ is at all points in the direction of ξ and we simply have

$$D_t\Gamma=\nu\int P\,da \ , \tag{A12}$$

which, using the results for P below, is

$$D_t\Gamma=-2\nu\pi r\left[\left[\frac{\partial w}{\partial r}\right](r)-\left\langle\frac{\partial w}{\partial r}\right\rangle_r\right]-\nu\pi w r^2c^2 \ , \tag{A13}$$

where $\langle\partial w/\partial r\rangle_r$ is the average vorticity gradient out to r, $[w(r)-w(0)]r$. A sharper than average gradient at r gives a flow outward of circulation. Curvature decreases circulation through a quadratic term.

In the general case we can use the identity

$$\Delta\boldsymbol{\omega}=-\nabla\times\nabla\times\boldsymbol{\omega}+\nabla(\nabla\cdot\boldsymbol{\omega}) \tag{A14}$$

and find that the dissipation is

$$D_t\Gamma=\nu\oint_C(\nabla\times\boldsymbol{\omega})\cdot dl \ . \tag{A15}$$

In the neck case dl is parallel to $\mathbf{b}$ and we find, by way of the result

$$\nabla\times\xi=c\mathbf{b} \ , \tag{A16}$$

true for neck vortices that

$$D_t\Gamma=-2\pi\nu r|(\mathbf{n}\cdot\nabla)w|\pm2\pi\nu rcw \ , \tag{A17}$$

where c is the curvature. The first term is the dissipation of circulation down the vorticity gradient out of the vortex. The second term represents the contribution of the curvature to the dissipation. It is only linear; we see that the effect of curvature is screened relative to the cylinder vortex case. The sign depends on the configuration—the curvature dissipates circulation out of a bulged vortex and into a pinched vortex.

We return to our analysis of P and R for these two cases. In the cylindrical case clearly

$$P=\Delta w-wc^2 \tag{A18}$$

and one can show that

$$\mathbf{R}=wcT\mathbf{b}+w[(\xi\cdot\nabla)c]\mathbf{n}+c[(\xi\cdot\nabla)w]\mathbf{n} \ , \tag{A19}$$

where T is the torsion. The viscosity tends to dissipate the vortex wherever curved. This same curvature creates a self-induced strain tending to amplify the vorticity. Since the strain is linear in the curvature, the dissipation should win in time for strong curvatures.

We see that we have just the same terms for **R** as in the filament case, in addition to a torsion term, since we are living in 3D space. One can easily see that this term will tend to "unwind" a helical twist in a vortex. On such a helical vortex, lying locally on a cylinder, the direction **b** is on the cylinder normal to the vortex. The new term tends to distend the vortex along the cylinder.

We now look directly at the equation for the evolution of the curvature of a vortex. Actually in general the effect of viscosity on the vortices is a little more complicated than presented up to this point. We described the viscous force **R** as bending a vortex by pushing it sidewise or, more precisely, by pushing it more than it is pushed nearby. However, the viscous part of the equation for the general evolution of the vortex curvature is

$$D_t c = \nu \mathbf{n}\cdot\left[(\xi\cdot\nabla)\left[\frac{\mathbf{R}}{w}\right]\right]+\frac{\nu R}{w}\mathbf{n}\cdot(\hat{\mathbf{R}}\cdot\nabla)\xi \ . \qquad \text{(A20)}$$

In the first term the displacement force $\mathbf{R}/w$ appears as an effective curvature component of the strain (although one that is a *local* function of the vorticity). If there is a gradient of this effective strain along the vortex line it will bend the line. However, we have an additional term, although it does not appear in the case of the perfectly aligned cylindrical vortex. It expresses the fact that due to the rotation of the vectors and the reconfiguration of the field into vortex lines, the vector may now be on a new line with different curvature.

Let us now substitute in our expression for R. The first gives

$$D_t c = \frac{\nu}{w}\left[-cT^2+c_{\xi\xi}+\frac{2}{w}c_\xi w_\xi+\frac{2c}{w}w_{\xi\xi}-\frac{2c}{w^2}w_\xi^2\right] , \qquad \text{(A21)}$$

where the ξ subscript means differentiation along the vortex (i.e., $\xi\cdot\nabla$), or

$$D_t c = \frac{\nu}{w}[-a(\xi)c+b(\xi)c_\xi+c_{\xi\xi}] , \qquad \text{(A22)}$$

where

$$a = T^2+2\left[\frac{w_\xi}{w}\right]^2-2\left[\frac{w_{\xi\xi}}{w}\right] \qquad \text{(A23)}$$

and

$$b = 2\frac{w_\xi}{w} \ . \qquad \text{(A24)}$$

If there is no variation of the vorticity magnitude along the vortex, we see that we have a diffusion of the curvature along the vortex, together with an exponential decay in curvature—straightening—in helical areas with a nonzero torsion. In a (small enough) neighborhood of a maximum in vorticity $(w_\xi/w)^2-2(w_{\xi\xi}/w)>0$ and this increases the vortex straightening. If the vorticity peak stems from self-induced strain due to a sharp curve in the vortex this will alleviate this nonlinear growth. The c_ξ term may cause waves of curvature along the vortex—this issue has not been investigated yet.

In the case of a neck vortex we have as we recall a ξ Laplacian term and a mixed w-ξ term. **R** is in the **n** direction because of the axisymmetric configuration. Note that **b** is the $\hat{\theta}$ direction in cylindrical coordinates. At the point considered, we call the distance from the vortex axis r and the angle of ξ from the z axis ψ. Each vortex line has curvature c and the lines diverge in the r-θ plane. This divergence is quantified by $\mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\xi$. Since this is the same as $-\xi\cdot(\mathbf{n}\cdot\nabla)\mathbf{n}$, which is the curvature of the **n** field, we label it c_n.

The symmetries of this case simplify the ξ Laplacian and the end result is that

$$\mathbf{R}_n = w\mathbf{n}\cdot\Delta\xi+2\mathbf{n}\cdot(\nabla w\cdot\nabla)\xi \qquad \text{(A25)}$$

$$= \mathbf{R}_{\text{line}}+\mathbf{R}_{\text{fil}}+\mathbf{R}_{3\text{D}} , \qquad \text{(A26)}$$

where

$$\mathbf{R}_{\text{line}} = w(\mathbf{n}\cdot\nabla)c_n+2[\mathbf{n}\cdot\nabla)w]c_n , \qquad \text{(A27)}$$

$$\mathbf{R}_{\text{fil}} = w\mathbf{n}\cdot(\xi\cdot\nabla)c+2[(\xi\cdot\nabla)w]c , \qquad \text{(A28)}$$

and

$$\mathbf{R}_{3\text{D}} = wc\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\xi+wc_n\mathbf{b}\cdot(\mathbf{b}\cdot\nabla)\mathbf{n}+w\mathbf{n}\cdot(\mathbf{b}\cdot\nabla)(\mathbf{b}\cdot\nabla)\xi \ . \qquad \text{(29)}$$

So again in the binormal frame we can see that the $\mathbf{R}_{\text{fil}}$ terms are just the terms we had in the equations for a filament in Sec. VI and as we have seen the tendency of these is to iron out the vortex filaments making up the vortex tube. The $\mathbf{R}_{\text{line}}$ terms just correspond to the terms in the equation for vectors on a line, here the vectors along the radius of the tube.

The remaining $\mathbf{R}_{3\text{D}}$ terms are different and exist only in three dimensions. From the fact that on a circle $\partial_\theta\hat{r}=1/r$, we can see that these terms are just

$$\mathbf{R}_{3\text{D}} = (\pm)^{p_\xi}\frac{\sin(\psi)}{r}c+(\pm)^{p_n}\frac{\cos(\psi)}{r}c_n +(\pm)^{p_n}(\pm)^{p_\xi}\frac{\cos(\psi)\sin(\psi)}{r^2} \ . \qquad \text{(A30)}$$

The signs p_ξ and p_n are determined by whether the vector fields ξ and **n** point outwards (+) or inwards (−) in the radial direction.

The last term is just the Laplacian in the axial direction. From examination it tends to act outwards on pinches and inwards on bulges in the tube and thus to straighten out the tube. One can understand this in terms of our 2D analysis even though this is a purely 3D phenomenon—it stems from the fact that the radial components of the vortex lines make up a ring of vectors whose value, as we saw in the 2D analysis, decays exponentially in time. Thus the radial component, the departure from a straight vortex, decays in time and the tube tends towards being straight.

The dissipation force P equals

$$P=(\nabla\nabla w)_{\xi\xi}+(\nabla\nabla w)_{nn}-wc^2-wc_n^2-w\frac{\sin^2(\psi)}{r^2}\ , \quad \text{(A31)}$$

where $(\nabla\nabla w)_{\xi\xi}=\sum\xi_i\xi_j\partial_i\partial_j w$ and likewise for the n-n component. The first two terms come from the Laplacian of the vorticity magnitude and the last three from the lack of alignment in the direction field. The last term, as we just saw, results from the curvature inherent in a circle of vectors, just like the straightening effect just mentioned.

The other two terms are single derivative terms resulting from spatial variation of the frame, here the binormal frame, or more precisely from derivatives of the scale factors of the frame. They are exactly analogous to the "extra" $(1/r)(\partial/\partial r)$ term in the Laplacian in cylindrical coordinates. Their effect on a vortex pinch is ambiguous in that they distort, but do not simply pull out or push in a pinch in a vortex.

In summary, along the vortex the viscosity tends to smooth out the vortex lines. On the plane normal to the vortex the viscosity will tend to align the vortex vectors parallel to the strongest filament in the vortex.

In conclusion we discuss briefly a third kind of effect, that of convection, on vortex lines. The effect of convection on vortex lines, being nonlocal, can be nontrivial in the sense of not being ignorable even in a moving frame.

We look first at the case of a neck vortex. Let us look at the velocity, at a point that we take as $\mathbf{x}=0$, as inducted by the vortex. This velocity is, by the Biot-Savart law,

$$\mathbf{u}=\frac{1}{2\pi}\int_{\text{vortex}}\frac{\mathbf{y}\times\boldsymbol{\omega}(\mathbf{y})}{y}d^3y\ . \quad \text{(A32)}$$

We use a frame where y_2 is in the direction of the closest point on the vortex core line, y_3 in the direction along the vortex, and y_1 in the transverse direction. Then the symmetries of the problem are

$$w_1(y_1)=-w_1(-y_1)\ , \quad \text{(A33)}$$

$$w_2(y_1)=w_2(-y_1)\ , \quad \text{(A34)}$$

and

$$w_3(y_1)=w_3(-y_1)\ . \quad \text{(A35)}$$

From this it is easy to see that the only velocity component is in the y_1 direction, i.e., in cylindrical coordinates we have $u_r=0$, $u_z=0$. This means that the vortex's self-inducted velocity alone cannot swell or pinch (or, in fact, stretch) the tube. However, the u_θ, if varying (as is very reasonable) along the undulating vortex, can twist the vortex lines into a helical configuration.

In the case of the cylinder vortex, convection can disturb vortex line alignment. For example, in this case we have seen that the leading term in the self-induced velocity in the center of the vortex, for vortices whose width is much smaller than their radius of curvature, is a velocity proportional to $\boldsymbol{\omega}$ and the curvature in the binormal direction. Points off the vortex center will have the same term but multiplied by some factor depending on the distance from the center, so the motion of the tube in this velocity deforms the vortex cross section. Finally, note that convective effects on the lines are inversely proportional to the scale of the vortex curvature and torsion and can be limited to this extent.

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